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THE METHOD OF VIRTUAL ABSORPTION IN PLANE DYNAMIC PROBLEMS*

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The method of virtual absorption /1/ is applied to plane dynamic contact problems of the theory of elasticity, and it modification in the part concerning the selection of basic functions is presented. It is shown that systems of delta functions with nonintersecting carriers may be used as basic functions. Results of numerical analysis of a concrete problem which show the effectiveness of the proposed method are presented. Other methods of solving these problems were used in /2—5/.

1. In a number of plane dynamic contact problems on the oscillation of stamps on the surface of regions whose boundaries stretch to infinity the integral equations are of the form

$$\int_{-a}^{a} k(x-\xi)\,q(\xi)\,d\xi = 2\pi f(x), \quad |x| \leqslant a, \qquad k(x) = \int_{\sigma} K(u)\,e^{-iux}\,du \tag{1.1}$$

where $q(x)$ are amplitudes of the unknown complex contact stresses and $f(x)$ are the amplitudes of functions that define the motion of the stamp surface.

The properties of function $K(u)$ are given in /6/. They comprise: analyticity, evenness, and realness for real arguments, and the representation of such functions in the form of the ratio of two entire functions whose behavior at infinity is defined by $c\,|u|^{-1}$, $|u| \to \infty$.

The selection of contour σ is dictated by the principle of virtual absorption in conformity with rules indicated in /6,7/.

According to /8/ the integral equation (1.1) is uniquely solvable for any twice continuously differentiable right-hand side in the space of functions that are continuous with weight, and the correction formula is

$$\| q(x)\sqrt{a^2-x^2} \|_C \leqslant N \| f \|_{C'}$$

For constructing an approximate solution of the integral equation (1.1) we approximate function $K(u)$ by the expression

$$K_1(u) = K_0(u)\,H(u), \quad H(u) = c\prod_{k=1}^{n} (u^2 - z_k^2)(u^2 - p_k^2)^{-1} \tag{1.2}$$

As $K_0(u)$ we can take $(u^2+B^2)^{-1/2}$ or $u^{-1}\operatorname{th} Au$, taking into account the properties of function $K(u)$ defined above and, also, the property

$$|K(u) - K_1(u)|\,|K^{-1}(u)|\,(1+|u|)^{\alpha} < \delta, \quad \alpha > 1/2, \quad |u| \leqslant \infty$$

By Theorem 2 in /6/ we have in this case also the closeness to solutions of integral equations of the form (1.1) with kernels $K(u)$ and $K_1(u)$, for fairly small δ. In the above equation $B > 0$ is an arbitrary parameter whose selection will be dealt with later.

For solving the integral equation (1.1) with kernel $K_1(u)$ we use the method developed in /1/.

We seek a solution of the form

$$q(x) = q_0(x) + \varphi(x) \tag{1.3}$$

where the unknown function $\varphi(x)$ is chosen on the basis of the condition of equality of functionals

$$\int_{-a}^{a} q(x)\,e^{\pm ip_k x}\,dx \equiv \int_{-a}^{a} \varphi(x)\,e^{\pm ip_k x}\,dx, \qquad k = 1, 2, \ldots, n \tag{1.4}$$

These functionals of the unknown function $q_0(x)$ are zero, and p_k are poles of function $H(u)$ such that $\operatorname{Im} p_k \geqslant 0$.

Application of the method of /1/ presupposes the expansion of function $\varphi(x)$ in any complete linearly independent system of functions. As such function we take the Dirac delta function with carriers at points x_k, i.e. we seek a solution of the form

$$q(x) = q_0(x) + \sum_{k=1}^{2n} c_k\delta(x - x_k), \quad |x_k| < a \tag{1.5}$$

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where c_k are constants that are to be determined and $x_k = \pm y_k$, y_k are points that divide the interval $(0, a)$ in three equal segments.

Without loss of generality, we set $f(x) = e^{-i\eta x}$, $\operatorname{Im}\eta = 0$. The substitution of (1.5) into (1.1) yields an equation of the form

$$\int_{-a}^{a} k_0(x-\xi)\, t(\xi)\, d\xi = g(x), \quad |x| \leqslant a, \qquad k_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_0(u)\, e^{-iux}\, du \tag{1.6}$$

$$g(x) = e^{-i\eta x} - \frac{1}{2\pi}\sum_{k=1}^{2n} c_k k(x - x_k) \tag{1.7}$$

$$T(u) = H(u)\, Q_0(u), \quad T(u) = \int_{-a}^{a} t(x)\, e^{iux}\, dx \tag{1.8}$$

Function $q_0(x)$ introduced by formula (1.3) is defined by the relation

$$q_0(x) = \frac{1}{2\pi}\int_\sigma Q_0(u)\, e^{-iux}\, du \tag{1.9}$$

Equation (1.6) with arbitrary right-hand side $g(x)$ is solve asymptotically for $K_0(u) = (u^2 + B^2)^{-1/2}$ /6/ and exactly for $K_0(u) = u^{-1}\operatorname{th} Au$ /9,10/.

We denote by $T(u, \eta)$ the Fourier transform of solution of the integral equation (1.6) with the right-hand side $g(x) = e^{-i\eta x}$. After the application of the Fourier transform the solution of Eq. (1.6) with the right-hand side (1.7) assumes the form

$$T(u) = T(u, \eta) - \frac{1}{2\pi}\sum_{k=1}^{2n} c_k \int_\sigma T(u, \eta)\, K(\eta)\, e^{i\eta x_k}\, d\eta$$

Let us consider the case when $K_0(u) = (u^2 + B^2)^{-1/2}$ and when in conformity with stipulations of the method of virtual absorption, we set $B \gg 1$. Using the methods of /6/ and taking into account that function $K_0(u)$ has no singularities on the real axis, we obtain

$$T(u, \eta) = i\,\frac{\sqrt{(B+iu)(B-i\eta)}}{\eta - u}\, e^{i(u-\eta)a} - i\,\frac{\sqrt{(B-iu)(B+i\eta)}}{\eta - u}\, e^{-i(u-\eta)a} + 2\pi\delta(u-\eta)\sqrt{u^2 + B^2}$$

Note that condition (1.4) is satisfied, since $Q_0(\pm p_k) = 0$, $k = 1, 2, \ldots, n$.

According to the lemma in /1/ for function $q_0(x)$ to have a carrier in region Ω $(\Omega: |x| \leqslant a)$ it is necessary and sufficient that

$$\int_{-a}^{a} g(x)\, e^{\pm iz_k x}\, dx = 0, \quad k = 1, 2, \ldots, n$$

from which we have

$$T(\pm z_k) = 0, \quad k = 1, 2, \ldots, n \tag{1.10}$$

where z_k are zeros of function $H(u)$ such that $\operatorname{Im} z_k \geqslant 0$.

Relations (1.10) represent $2n$ equalities for the determination of $2n$ unknown c_k, which for the determination of coefficients c_k reduce to the system

$$\sum_{k=1}^{2n} c_k A(\pm z_l, x_k) = B(\pm z_l, \eta), \quad l = 1, 2, \ldots, n, \quad A(u, x_k) = f(u)\, F(u, x_k) + f(-u)\, F(-u, -x_k), \tag{1.11}$$

$$f(u) = \sqrt{B - iu}\, e^{-iau}, \quad F(u, x_k) = \sum_{m=1}^{n} \frac{D_m e^{i(a+x_k)p_m}}{\sqrt{B - ip_m}\,(p_m - u)}, \quad D_m = \operatorname*{Res}_{u\to p_m} H(u)$$

$$B(u, \eta) = i(u-\eta)^{-1}[f(-\eta)\, f(u)\, \varphi(u) - f(\eta)\, f(-u)\, \varphi(-u) -$$

$$\sqrt{\eta^2 + B^2}\,(\varphi(\eta) - 1)\, e^{ia(u-\eta)} + \sqrt{\eta^2 + B^2}\,(\varphi(-\eta) - 1) e^{-ia(u-\eta)}], \quad \varphi(u) = \operatorname{erf}\sqrt{2a(B - iu)}$$

Having determined c_k using system (1.11) we determine function $q_0(x)$ by formulas (1.8) and (1.9). We have

$$q_0(x) = \frac{1}{2\pi}\int_\sigma \frac{T(u,\eta)}{H(u)}\, e^{-iux}\, du - \frac{1}{2\pi}\sum_{k=1}^{2n} c_k \int_\sigma \frac{A(u, x_k)}{H(u)}\, e^{-iux}\, du - \sum_{k=1}^{2n} c_k \delta(x - x_k)$$

Taking into account (1.5) we obtain the contact stresses under the stamp

$$q(x) = \frac{e^{-i\eta x}}{K(\eta)} + \frac{\sqrt{B - i\eta}}{2\pi i}\, e^{-ia\eta}\int_\sigma \frac{\sqrt{B + iu}}{H(u)(u-\eta)}\, e^{i(a-x)u}\, du - \frac{\sqrt{B + i\eta}}{2\pi i}\, e^{ia\eta}\int_\sigma \frac{\sqrt{B - iu}}{H(u)(u-\eta)}\, e^{-i(a+x)u}\, du - \tag{1.12}$$

$$\frac{1}{2\pi}\sum_{k=1}^{2n} c_k \int_\sigma \frac{\sqrt{B-iu}}{H(u)}\left[F(u, x_k)\, e^{-i(a+x)u} + F(u, -x_k)\, e^{-i(a-x)u}\right] du$$

Function $H(u)$ can be represented in the form

$$H(u) = [1 + H_1(u)]\, c, \quad H^{-1}(u) = [1 + H_2(u)]c^{-1}, \quad H_1(u) = \sum_{i=1}^{n} \alpha_i (u^2 - p_i^2)^{-1}, \quad H_2(u) = \sum_{j=1}^{n} \beta_j (u^2 - z_j^2)^{-1}$$

$$\alpha_i = \prod_{k=1}^{n} (p_i^2 - z_k^2) \prod_{\substack{k=1\\ i\neq k}}^{n} (p_i^2 - p_k^2)^{-1}, \qquad \beta_j = \prod_{k=1}^{n} (z_j^2 - p_k^2) \prod_{\substack{k=1\\ j\neq k}}^{n} (z_j^2 - z_k^2)^{-1}$$

Using these relationships and applying formulas of operational calculus to (1.12) we obtain

$$q(x) = \frac{e^{-i\eta x}}{K(\eta)} + \frac{e^{-B(a-x)}}{c\sqrt{\pi(a-x)}}\sqrt{B - i\eta}\, e^{-ia\eta} + \frac{e^{-B(a+x)}}{c\sqrt{\pi(a+x)}}\sqrt{B + i\eta}\, e^{ia\eta} + \frac{e^{-i\eta x}}{K(\eta)}\left[\operatorname{erf}\sqrt{(B+i\eta)(a-x)} + \right. \tag{1.13}$$

$$\left.\operatorname{erf}\sqrt{(B-i\eta)(a+x)} - 2\right] + \frac{1}{c}\sum_{i=1}^{n} \times \frac{\beta_i}{2z_i}\left[\sqrt{B-i\eta}\, e^{-ia\eta}\Phi_i(\eta, x) + \sqrt{B+i\eta}\, e^{ia\eta}\Phi_i(-\eta, -x)\right] -$$

$$i\frac{e^{-B(a-x)}}{\sqrt{\pi(a-x)}}\sum_{k=1}^{2n} c_k \sum_{j=1}^{n} \frac{\alpha_j}{2p_j} F_j(x_k) - i\frac{e^{-B(a+x)}}{\sqrt{\pi(a+x)}}\sum_{k=1}^{2n} c_k \sum_{j=1}^{n}\frac{\alpha_j}{2p_j} F_j(-x_k) -$$

$$i\sum_{k=1}^{2n} c_k \sum_{j=1}^{n} \frac{\alpha_j}{2p_j}\sum_{i=1}^{n}\frac{\beta_i}{2z_i}\left[F_j(x_k)\,\Phi_i(-p_i, x) + F_j(-x_k)\,\Phi_i(-p_j, -x)\right]$$

Functions $F_j(x_k)$ and $\Phi_i(\eta, x)$ are of the form

$$F_j(x_k) = \frac{e^{ip_j(a-x_k)}}{\sqrt{B - ip_j}}$$

$$\Phi_i(\eta, x) = \frac{\sqrt{B+iz_i}}{z_i - \eta}\, e^{iz_i(a-x)} \operatorname{erf}\sqrt{(B+iz_i)(a-x)} + \frac{\sqrt{B - iz_i}}{z_i + \eta}\, e^{-iz_i(a-x)}\left[\operatorname{erf}\sqrt{(B-iz_i)(a-x)} - 1\right]$$

The contact stresses $q(x)$ were computed by formula (1.13) on a computer.

2. As an example we shall consider the plane problem or vibration of a stamp of width $2a$ resting without friction on an elastic layer of thickness h rigidly coupled to an undeformed base. The problem is reduced to an integral equation of the form (1.1).

When the specified vertical displacements of the stamp at the layer surface in region $x \in [-a, a]$ are harmonic, the kernel $K(u)$ is of the form

$$K(u) = {}^1/_4\varkappa_2^2\sigma_1(\sigma_1\sigma_2 \operatorname{sh} h\sigma_1 \operatorname{ch} h\sigma_2 - u^2 \operatorname{ch} h\sigma_1 \operatorname{sh} h\sigma_2)\{(2u^4 - u^2\varkappa_2^2)\sigma_1\sigma_2 - (2u^4 -$$
$$u^2\varkappa_2^2 + {}^1/_4\varkappa_2^4)\sigma_1\sigma_2 \operatorname{ch} h\sigma_1 \operatorname{ch} h\sigma_2 + u^2[2u^4 - u^2(\varkappa_1^2 + 2\varkappa_2^2) + {}^1/_4\varkappa_2^4 + \varkappa_1^2\varkappa_2^2] \operatorname{sh} h\sigma_1 \operatorname{sh} h\sigma_2\}^{-1}$$

$$\sigma_k = \sqrt{u^2 - \varkappa_k^2}, \quad \operatorname{Re}\sigma_k \geqslant 0, \quad \operatorname{Im}\sigma_k \leqslant 0, \quad k = 1, 2$$
$$\varkappa_1^2 = \frac{\rho\omega^2}{\lambda + 2\mu}, \quad \varkappa_2^2 = \frac{\rho\omega^2}{\mu}, \quad \varkappa_1^2 = \frac{1-2\nu}{2(1-\nu)}\varkappa_2^2$$
$$K(u) \to c\,|u|^{-1}, \quad |u| \to \infty, \quad c = 1 - \nu$$

where λ and μ are Lamé coefficients, and $\nu, \rho,$ and ω are, respectively, the Poisson coefficient, density of material, and the stamp oscillation frequency. Function $K(u)$ has all the properties ennumerated above and is approximated by function $K_1(u)$ of form (1.2).

To construct the approximating function we first determine the curves of real zeros and poles of kernel $K(u)$ in terms of parameter $\varkappa_2$. Then, using Bernstein or Lagrange polynomials, the approximation of the rational function (1.2) is obtained, reaching in this way the a priori specified accuracy. The approximation is determined on a computer.

The selection of parameter B as large as possible is dictated by the desire to devise effective solution approximations, however excessively large values of B result in an increased approximation error or a sharp increase of the order of approximating polynomials. In our numerical calculations we assumed $B = 10$. Formula (1.13) was used for calculating contact stresses on a computer. The change of stresses with increasing generalized frequency $\varkappa_2$ was observed in the case of the boundary problem formulated above.

Curves of the real part of $q(x)$ are shown in Fig.1 for several values of parameter $\varkappa_2$, with $\eta = -1$ and $a = 5$. Curves of the real (solid line) and imaginary (dash line) parts of $q(x)$ are shown in Fig.2 for $\varkappa_2 = 2.6$ and $a = 3$, and in Fig.3 for $\varkappa_2 = 3.4$ and $a = 5$, with $\eta = 0, -0.5,$ and -1.

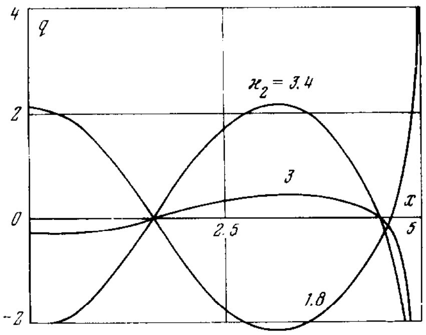

Fig.1

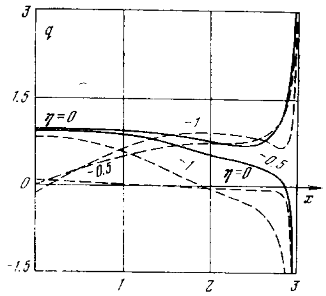

Fig.2

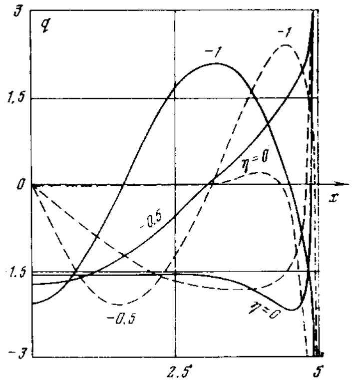

Fig.3

The obtained data are in good agreement with those of numerical calculations in /11/. However, the method presented here is more effective than that used in /11/, since it makes possible the allowance for the singularity at the stamp edge. The curves which define the behavior of the real part of $q(x)$ (Fig.1) coincide with the curves in /11/ throughout the interval $(-a, a)$ except at its ends.

It should be pointed out that the proposed method is applicable to stamps of large and small width and, also, to three-dimensional problems.

3. This method proved to be very effective in a number of three-dimensional problems, such as vibration of a stamp of wedge-shaped cross section, or of dynamics of a wedge shaped cavity.

Application of the Mellin transform over the radius issuing from the wedge apex reduces the integral equation of the three-dimensional problem to the equation of the plane problem of the form (1.1), and this makes possible the use of the proposed here method. It follows from Sect. 1 that one of the important aspects is the derivation of solution $q_0(x)$ for some limit equation (1.6).

The kernel of the contact problem on the action of a stamp of wedge-shaped cross section on an elastic half-space /12/ reduced to the form

$$k_0(s,\varphi) = 2\sum_{k=1}^{\infty} a_k(s)\cos k\varphi + a_0\,,$$

$$a_k(s) = \frac{\Gamma\left(\frac{s}{2}+\frac{k}{2}+\frac{1}{4}\right)\Gamma\left(-\frac{s}{2}+\frac{k}{2}+\frac{1}{4}\right)}{2\Gamma\left(\frac{s}{2}+\frac{k}{2}+\frac{3}{4}\right)\Gamma\left(-\frac{s}{2}+\frac{k}{2}+\frac{3}{4}\right)}, \quad k=0,1,2,\ldots$$

is the kernel of the limit equation in the problem of vibration of a wedge-shaped stamp. In the above equation s is the parameter of the Mellin transform and $\Gamma(x)$ is the Euler function.

In Eq. (1.1) $a=\alpha$ and 2α is the wedge apex angle.

The approximate solution of this integral equation which is valid for all angles α can be represented in a form different from that in /14/ by applying the method of /13/. That solution is of the form

$$q_0(s) = t(\varphi,0) - \sum_{k=1}^{\infty} c_k(s)\left[t(\varphi,k) + t(-\varphi,k)\right] \tag{3.1}$$

where $t(\varphi,0) = [2\beta(s)]^{-1}[R(e^{i\varphi}) + R(e^{-i\varphi})]$

$$t(\varphi,k) = [2\beta(s)]^{-1}(1 + {}^1/_2P_k + {}^1/_2P_{k-1})[R(e^{i\varphi}) + R(e^{-i\varphi})] +$$

$$\frac{k}{2}[P_kR(e^{-i\varphi}) - P_{k-1}R(e^{i\varphi})] + \frac{k^2}{4}\sum_{\substack{m=-\infty\\ m\neq k}}^{\infty}(P_kP_{m-1} - P_{k-1}P_m)\frac{e^{im\varphi}}{m-k} +$$

$$\frac{k}{2}e^{ik\varphi}\left[1-\cos\alpha + \sum_{m=2}^{k}(P_m - 2\cos\alpha P_{m-1} + P_{m-2})P_{m-1}\right], \quad k\neq 0$$

$$R(e^{i\varphi}) = \frac{e^{-i\varphi/2}}{\sqrt{2(\cos\varphi-\cos\alpha)}}, \quad \beta(s) = a_0 - \ln\frac{1-\cos\alpha}{2}$$

and $P_k = P_k(\cos\alpha)$ are Legendre polynomials.

Coefficients c_k are determined by the system

$$c_m(s) = b_m(s)\,t_m(0) - b_m(s)\sum_{k=1}^{\infty}c_k[t_m(k) + t_m(-k)], \quad m=1,2,\ldots; \qquad b_k(s) = a_k(s) - \frac{1}{k}, \quad k=1,2,\ldots \tag{3.2}$$

where t_m are the Fourier coefficients of function $t(\varphi, k)$.

Inversion of the Mellin transform by the method of residues, when only zeros s_k of the determinant $\Delta(s)$ of system (3.2) and functions $\beta(s)$ are known, results for any α in a solution of the problem of the form

$$q_0(r, \varphi) = \sum_{k=0}^{\infty} \theta(s_k, \varphi) r^{-s_k-3/2}, \quad \mathrm{Re}\,|s_{k+1}| < \mathrm{Re}\,|s_k| \tag{3.3}$$

The approximate values of s_0^* for $\beta(s_0) = 0$ and $\Delta(s_0) = 0$ are the same.

The dependence of s_0^* on a α is in good agreement with that first obtained by Rvachev /14/ and is of the form

α	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
s_0^*	−0.5	−0.76	−0.63	−1.0	−0.37	−0.24	−1.5

The system of integral equations for the determination of normal and tangential stresses for a wedge-shaped rigidly coupled with an elastic medium, reduces for small apex angles α to two separate equations for the form (1.1).

For small α the kernels of differential equations are

$$K_{1,2}(\varphi) = \ln \frac{1}{2\sin \frac{|\varphi|}{2}} \mp \frac{i\pi(1-2\nu)}{4(1-\nu)} \operatorname{sign} \varphi, \quad |\varphi| \leqslant \alpha$$

which were investigated in /15/.

The behavior of these stresses at the stamp apex is defined by a formula similar to (3.3) with

$$s_0 = -\frac{1}{2} - \frac{\varepsilon}{2(1-\nu)} + O(\varepsilon^2), \quad \varepsilon = -\frac{1}{\ln \alpha}, \quad \alpha \to 0$$

The remainder term contains the imaginary component.

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